Orbit Determination Singularities in the Doppler Tracking of a Planetary Orbiter

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The problem of determining the orbit of a spacecraft about another planet (or an asteroid or cometary nucleus) by means of Doppler tracking data is treated. The Doppler shift associated with the relative motion of an orbiting spacecraft and an Earth-based tracking station is derived mathematically and expressed as a power-series expansion in several small quantities. Partial derivatives of the Doppler shift with respect to the orbital elements and the mass of the attracting body are evaluated analytically, assuming simple two-body motion. Linear dependencies among the partial derivatives indicate unobservable parameters. Eight distinct cases are investigated, according to whether the orbit is elliptical or circular, whether the orbit is viewed face-on or at some other angle, and whether the central-body mass is known a priori or unknown. The full set of singular geometries for the determination of the orbit of a spacecraft about another planet, using Doppler data, is justified mathematically, for the first time. Certain geometries previously hypothesized to be singular are shown to be nonsingular.

Introduction

On a number of occasions during the 1960s and 1970s, spacecraft launched by the United States have been placed into orbit about the moon, Venus, or Mars. The lunar missions of this type include Lunar Orbiters 1-5 and Apollos 8, 10-12, and 14-17. The planetary orbiter missions include Mariner 9 and Vikings 1 and 2 to Mars, as well as the Pioneer Venus Orbiter. In the planetary orbiter missions, in particular, the principal data type for orbit determination purposes has been two-way coherent Doppler data.[1-4]

It has long been known, based on analytical studies and numerical simulations, as well as the processing of flight data, that some orbital elements are much more easily determined by Doppler data than other orbital elements, in the case of a planetary or lunar orbiter.[2-14] In addition, it has long been known that orbit determination accuracies are strongly dependent on orbital geometry, with certain geometries, sometimes referred to as "singular," producing near indeterminacies in some orbital elements.

The purpose of this paper is to derive, within a consistent mathematical framework, all the geometrical conditions that will cause problems in determining some orbital elements. Important or interesting special cases of general elliptical orbits, such as circular orbits and orbits viewed face on, are included. Situations with an accurately known central-body mass (typical of lunar or planetary orbiters) are treated, as well as situations with an unknown central-body mass (typical of an orbiter of an asteroid or a cometary nucleus). Some of these geometrical conditions are fairly obvious and are easily interpreted. Other conditions are far less obvious and have been misinterpreted at times in the past.

Among previous general studies of the determination of planetary or lunar orbits using Doppler data, Refs. 6, 8, and 11 are particularly pertinent to the development to follow. Reference 6 includes a qualitative discussion of which orbital elements can be determined for general elliptical orbits, based on an analytical expression for the Doppler shift similar to one derived below, but does not investigate problematical geometries. Reference 8 is a systematic study of the effect of orbit size, shape, and geometry on orbit determination accuracy by means of numerical simulation. This paper yields substantial insight into the importance of orbit geometry on orbit determination accuracy. However, since only one orbital element is varied at a time from a reference set of elliptical orbit elements, many possible orbit geometries remain uninvestigated. Reference 11 derives approximate analytical expressions for orbit determination uncertainties for the special case of a circular orbit. The problematical geometries for a circular orbit differ quite substantially from those for an elliptical orbit[10,11,13]—a fact that has caused considerable confusion over the years.

Interest in the accurate determination of the orbit of a spacecraft about a natural body other than the Earth (or the sun) is quite high at the present time, due to the large number of orbiter missions already approved or under consideration within the National Aeronautics and Space Administration for launches within the next twenty years. These missions include the following: Galileo, Venus Radar Mapper, Mars Observer, Comet Rendezvous/Asteroid Flyby, Lunar Geoscience Orbiter, Near-Earth Asteroid Rendezvous, Main-Belt Asteroid Rendezvous, Saturn Orbiter/Titan Probe, Mars Aeronomy Orbiter, Mars Sample Return, Comet Nucleus Sample Return, and Uranus and Neptune Orbiters.[14-17]

Doppler Shift Associated with a Planetary Orbiter

To determine the Doppler shift associated with the motion of an orbiting spacecraft, as seen from a tracking station on the Earth, we will need to determine the relative position and velocity vectors of the two objects and then the component of the latter vector in the direction of the former. Let r and v denote the position and velocity vectors of the center of mass of the planet being orbited relative to the tracking station. Let r_ω and v_ω denote the position and velocity vectors of the orbiting spacecraft relative to the center of mass of the planet. (The object being orbited could equally well be a planetary satellite, an asteroid, or a cometary nucleus; but, for brevity, the object being orbited will henceforth be referred to as a planet.) The line-of-sight velocity we seek is given by

$$v_{los} = (v+v_\omega)\cdot(r+r_\omega)/[(r+r_\omega)\cdot(r+r_\omega)]^{1/2} \qquad (1)$$

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To evaluate Eq. (1) in a useful fashion and to do subsequent calculations, it is helpful to introduce several coordinate systems, which are determined from four fundamental unit vectors, $\hat{x}_\omega, \hat{z}_\omega, \hat{z}$, and $\hat{z}_E$. $\hat{x}_\omega$ points from the center of mass of the planet being orbited toward periapsis; $\hat{z}_\omega$ points along the spacecraft orbital angular momentum vector (i.e., normal to the orbit plane); $\hat{z}$ points from the center of mass of the planet toward the tracking station on the Earth; and $\hat{z}_E$ points along the Earth's spin axis (positive northward). Assuming simple Keplerian motion of the spacecraft about the planet, $\hat{x}_\omega$ and $\hat{z}_\omega$ are orthogonal and are fixed in inertial space. $\hat{z}_E$ is also assumed fixed in inertial space. $\hat{z}$ is not fixed, due to the relative motion of the tracking station and the planet.

To establish the plane-of-sky coordinate system, define $\hat{x}$ to be a unit vector in the direction of $\hat{z}_E \times \hat{z}$ (the decreasing right ascension direction). This definition is basically arbitrary but happens to be convenient. Now, define the unit vectors $\hat{y}$ (in the direction of increasing declination) and $\hat{y}_\omega$ such that $\{\hat{x}, \hat{y}, \hat{z}\}$ and $\{\hat{x}_\omega, \hat{y}_\omega, \hat{z}_\omega\}$ are right-handed, orthonormal sets of vectors. We wish to relate these sets of vectors to one another by a sequence of Euler-angle rotations.

Define $\hat{x}'$ to be a unit vector in the direction of $\hat{z} \times \hat{z}_\omega$ (the line of nodes in the plane of the sky). Let $\hat{z}'$ be the same as $\hat{z}$, and choose the unit vector $\hat{y}'$ such that $\{\hat{x}', \hat{y}', \hat{z}'\}$ form a right-handed orthonormal set. Since $\hat{x}$ and $\hat{x}'$ are orthogonal to $\hat{z}$, it follows that a rotation of the $\hat{x}$-$\hat{y}$-$\hat{z}$ coordinate system about $\hat{z}$ through some angle Ω (the longitude of the ascending node in the planet-of-sky system) will cause $\hat{x}$ and $\hat{x}'$ (and $\hat{y}$ and $\hat{y}'$) to align. Thus,

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} c\Omega & s\Omega & 0 \\ -s\Omega & c\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{2}$$

where c and s denote cosine and sine. These unit vectors, and others to follow, are illustrated in Fig. 1.

Next, define $\hat{x}''$ to be the same as $\hat{x}'$ and $\hat{z}''$ to be the same as $\hat{z}_\omega$. Choose the unit vector $\hat{y}''$ such that $\{\hat{x}'', \hat{y}'', \hat{z}''\}$ form a right-handed orthonormal set. Since $\hat{z}'$ and $\hat{z}''$ are both orthogonal to $\hat{x}'$, it follows that a rotation of the $\hat{x}'$-$\hat{y}'$-$\hat{z}'$ coordinate system about $\hat{x}'$ through some angle i (the orbit inclination in the plane-of-sky system) will cause $\hat{z}'$ and $\hat{z}''$ (and $\hat{y}'$ and $\hat{y}''$) to align. Thus,

$$\begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & ci & si \\ 0 & -si & ci \end{bmatrix} \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} \tag{3}$$

Finally, a rotation of the $\hat{x}''$-$\hat{y}''$-$\hat{z}''$ coordinate system about $\hat{z}''$ through some angle ω (the argument of periapsis in the plane-of-sky system) will cause $\hat{x}''$ and $\hat{x}_\omega$ (and $\hat{y}''$ and $\hat{y}_\omega$) to align. Thus,

$$\begin{bmatrix} \hat{x}_\omega \\ \hat{y}_\omega \\ \hat{z}_\omega \end{bmatrix} = \begin{bmatrix} c\omega & s\omega & 0 \\ -s\omega & c\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} \tag{4}$$

Combining Eqs. (2-4), we have[18]

$$\begin{bmatrix} \hat{x}_\omega \\ \hat{y}_\omega \\ \hat{z}_\omega \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{5}$$

where

$$T_{11} = c\omega c\Omega - s\omega c i s\Omega \tag{6}$$

$$T_{12} = c\omega s\Omega + s\omega c i c\Omega \tag{7}$$

$$T_{13} = s\omega s i \tag{8}$$

$$T_{21} = -s\omega c\Omega - c\omega c i s\Omega \tag{9}$$

$$T_{22} = -s\omega s\Omega + c\omega c i c\Omega \tag{10}$$

$$T_{23} = c\omega s i \tag{11}$$

Expressions for T_{31}, T_{32}, and T_{33} are not needed in developments that follow.

In order to obtain tractable analytical expressions for subsequent use, we shall neglect all forces acting on the spacecraft except the point-mass gravitational attraction of the central body. Thus, the position and velocity of the spacecraft relative to the planet being orbited are given by[19]

$$r_\omega = a[(cE - e)\hat{x}_\omega + (1 - e^2)^{1/2} sE\hat{y}_\omega] \tag{12}$$

$$v_\omega = (\mu/a)^{1/2}[-sE\hat{x}_\omega + (1 - e^2)^{1/2} cE\hat{y}_\omega]/(1 - ecE) \tag{13}$$

and μ, a, e, and E denote the gravitational parameter of the planet, the orbital semimajor axis, the eccentricity, and the eccentric anomaly, respectively. (Eccentric anomaly rather than true anomaly is used here because it can be more easily related to time past periapsis.)

The position and velocity of the center of mass of the planet relative to the tracking station are given by

$$r = -R\hat{z} \tag{14}$$

$$v = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \tag{15}$$

where R is the distance between the tracking station and the center of mass of the planet being orbited and v_x, v_y, and v_z are simply the components of v resolved along the axes $\hat{x}$, $\hat{y}$, and $\hat{z}$.

We can now evaluate the various terms in the numerator and denominator of the right-hand side of Eq. (1). From Eqs. (14) and (15), we have

$$v \cdot r = -Rv_z \tag{16}$$

From Eqs. (12) and (13), we have

$$v_\omega \cdot r_\omega = (\mu a)^{1/2} esE \tag{17}$$

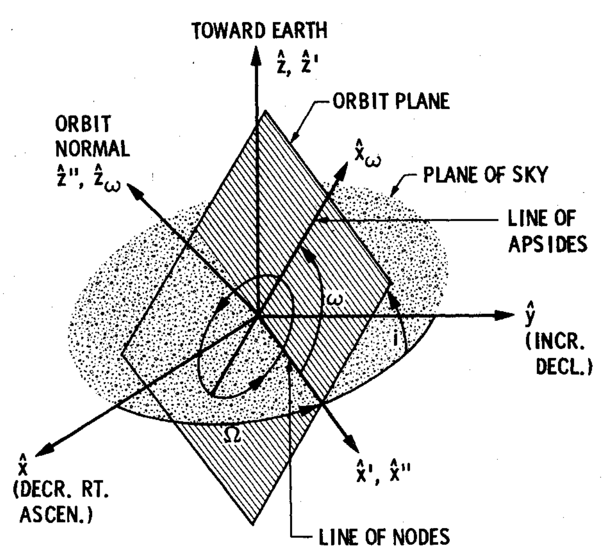

Fig. 1 Plane-of-sky geometry.

From Eqs. (5), (12), and (15), we have

$$v \cdot r_\omega = a[(cE - e)(T_{11}v_x + T_{12}v_y + T_{13}v_z) + (1-e^2)^{1/2} sE(T_{21}v_x + T_{22}v_y + T_{23}v_z)] \quad (18)$$

From Eqs. (5), (13), and (14), we have

$$v_\omega \cdot r = -R(\mu/a)^{1/2}[-sET_{13} + (1-e^2)^{1/2} cET_{23}]/(1-ecE) \quad (19)$$

Now, let us assume that

$$a/R \ll 1 \quad (20)$$

We find from Eqs. (5), (12), and (14) that

$$[(r+r_\omega)\cdot(r+r_\omega)]^{1/2} = R\{1 - (a/R)[(cE-e)T_{13} + (1-e^2)^{1/2} sET_{23}]\} \quad (21)$$

where terms inside the braces that are higher than first order in a/R have been neglected.

From Eqs. (1), (16-19), and (21), we obtain the result

$$v_{\text{los}} = -\{[na/(1-ecE)][-sET_{13} + (1-e^2)^{1/2} cET_{23}] + v_z\} \cdot \{1 + (a/R)[(cE-e)T_{13} + (1-e^2)^{1/2} sET_{23}]\} + (na^2/R)esE + (a/R)[(cE-e)(T_{11}v_x + T_{12}v_y + T_{13}v_z) + (1-e^2)^{1/2} sE(T_{21}v_x + T_{22}v_y + T_{23}v_z)] \quad (22)$$

where

$$n = (\mu/a^3)^{1/2} \quad (23)$$

Equation (22) gives the instantaneous line-of-sight velocity in terms of the instantaneous Euler angles Ω, i, and ω. These angles are not constant, however, and since we will be interested in data accumulated over an entire spacecraft orbit, their time variations will turn out to be important. To determine how these angles vary with time, note that the $\hat{x}$-$\hat{y}$-$\hat{z}$ coordinate system rotates with respect to inertial space at an angular velocity ξ such that the time rate of change of the vector r, in a nonrotating frame, is given by v. The time rate of change of r in the rotating $\hat{x}$-$\hat{y}$-$\hat{z}$ frame is $-(\mathrm{d}R/\mathrm{d}t)\hat{z}$. Thus, we have

$$v = -\frac{\mathrm{d}R}{\mathrm{d}t}\hat{z} + \xi \times r \quad (24)$$

With the use of Eqs. (14), (15), and (24), we conclude that

$$\xi = \frac{v_y}{R}\hat{x} - \frac{v_x}{R}\hat{y} + \xi_z \hat{z} \quad (25)$$

ξ_z is not yet known, but can be determined by requiring that $\hat{x}$ evolve with time so as to lie always in the direction of $\hat{z}_E \times \hat{z}$. It turns out that

$$\xi_z = v_x \frac{\tan\delta}{R} \quad (26)$$

where δ is the declination of the planet, as viewed from the tracking station.

Rather than express ξ in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$, as in Eq. (25), it is more convenient to express this angular velocity vector in terms of the nonorthogonal set of unit vectors $\{\hat{z}, \hat{x}', \hat{z}''\}$. From Eqs. (2), (25), and (26), then, we have

$$\xi = (v_t/R)[s\beta(c\Omega\hat{x}' - s\Omega\hat{y}') - c\beta(s\Omega\hat{x}' + c\Omega\hat{y}') + c\beta\tan\delta\hat{z}] = -(v_t/R)[s\Omega'\hat{x}' + c\Omega'\hat{y}' - c\beta\tan\delta\hat{z}] \quad (27)$$

where v_t (the transverse velocity) and Ω' (the angle between the line of nodes and the projection of the velocity vector v onto the plane of the sky) are defined according to

$$v_t = (v_x^2 + v_y^2)^{1/2} \quad (28)$$

$$v_x = v_t c\beta \quad (29)$$

$$v_y = v_t s\beta \quad (30)$$

$$\Omega' = \Omega - \beta \quad (31)$$

From Eqs. (2) and (3) we find that

$$si\hat{y}' = -\hat{z}'' + ci\hat{z} \quad (32)$$

Thus,

$$\xi = -(v_t/R)[s\Omega'\hat{x}' - (c\Omega'/si)\hat{z}'' + (\cot i c\Omega' - \tan\delta c\beta)\hat{z}] \quad (33)$$

The angular velocity of the nonrotating $\hat{x}_\omega - \hat{y}_\omega - \hat{z}_\omega$ coordinate frame with respect to the rotating $\hat{x}$-$\hat{y}$-$\hat{z}$ coordinate frame is simply $-\xi$ and is expressible in terms of time derivatives of Euler angles as

$$-\xi = \left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)\hat{z} + \left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)\hat{x}' + \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)\hat{z}'' \quad (34)$$

Consistency of Eqs. (33) and (34) then yields

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{v_t}{R}(\cot i c\Omega' - \tan\delta c\beta) \quad (35)$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{v_t}{R} s\Omega' \quad (36)$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{v_t}{R}\frac{c\Omega'}{si} \quad (37)$$

Now, let us return to Eq. (22) and replace Ω, i, and ω with first-order Taylor series expansions about $t = t_p^*$, where t_p^* denotes the nominal time of periapsis. Thus, we make the substitutions

$$\Omega \to \Omega + \frac{\mathrm{d}\Omega}{\mathrm{d}t}\Delta t \quad (38)$$

$$i \to i + \frac{\mathrm{d}i}{\mathrm{d}t}\Delta t \quad (39)$$

$$\omega \to \omega + \frac{\mathrm{d}\omega}{\mathrm{d}t}\Delta t \quad (40)$$

where Ω, i, and ω henceforth denote the values of the Euler angles at $t = t_p^*$ and

$$\Delta t = t - t_p^* = t - t_p + t_p - t_p^* \quad (41)$$

where t_p denotes the actual time of periapsis. If terms that are higher than first order in the small quantities a/R and $v_t\Delta t/R$ are neglected, Eq. (22) becomes

$$v_{\text{los}} = v_{\text{los}}^0 + v_{\text{los}}^1 \quad (42)$$

v_{los}^0 denotes the sum of terms of zeroth order in the small quantities and is given by

$$v_{\text{los}}^0 = -np_1 f_1(e, \omega, E) - v_z \quad (43)$$

where

$$p_1 = asi \tag{44}$$

$$f_1(e,\omega,E) = [-s\omega sE + (1-e^2)^{1/2} c\omega cE]/(1-ecE) \tag{45}$$

$v_{\rm los}^1$ denotes the sum of first-order terms and is given by

$$\begin{aligned} v_{\rm los}^1 &= v_{\rm los}^0 (p_1/R)[s\omega(cE-e) + (1-e^2)^{1/2} c\omega sE] \\ &+ (na^2/R)esE + (a/R)\{[v_t(c\omega c\Omega' - s\omega ci s\Omega') \\ &+ v_z s\omega si](cE-e) + (1-e^2)^{1/2}[v_t(-s\omega c\Omega' \\ &- c\omega ci s\Omega') + v_z c\omega si]sE\} + \{(v_t/R)[(\partial v_{\rm los}^0/\partial i)s\Omega' \\ &- (\partial v_{\rm los}^0/\partial\omega)(c\Omega'/si)] - {\rm d}v_z/{\rm d}t\}\Delta t \end{aligned} \tag{46}$$

Note that Eq. (35) was not needed in obtaining Eqs. (43) and (46). Note also that v_x, v_y, and v_z are not constant quantities, and that Eqs. (38-40), (43), and (46) were obtained by assuming that

$$\left|\frac{{\rm d}v_x}{{\rm d}t}\Delta t\right| \ll v_t \tag{47}$$

$$\left|\frac{{\rm d}v_y}{{\rm d}t}\Delta t\right| \ll v_t \tag{48}$$

$$\left|\frac{{\rm d}^2 v_z}{{\rm d}t^2}(\Delta t)^2\right| \ll \left|\frac{{\rm d}v_z}{{\rm d}t}(\Delta t)\right| \ll |v_z| \tag{49}$$

These assumptions are often satisfied in practice but may be less generally valid than the earlier assumptions about smallness of a/R and $v_t\Delta t/R$. Note that the motion of the tracking station due to the Earth's rotation, of great importance in determining the heliocentric motion of an interplanetary spacecraft,[1] is of relatively little significance in the planetary orbiter problem.

Equations (43-45) essentially duplicate an expression in Ref. 6. Some of the terms in Eq. (46) are missing in the corresponding expression in Ref. 6, however. This difference is due to the fact that $v_{\rm los}$ was not defined according to Eq. (1) in Ref. 6. Instead, $v_{\rm los}$ was defined to be the less exact quantity $\boldsymbol{v}_\omega \cdot \hat{\boldsymbol{z}}$. (The missing terms have little impact on the arguments that follow, however.)

Orbit Determination Using a Zeroth-Order Model for the Doppler Shift

Elliptical Orbits with Unknown Central-Body Mass

We wish to determine the seven quantities e, ω, t_p, n, a, i, and Ω' from measurements of $v_{\rm los}$ over the course of a spacecraft orbit. We shall first see what information can be obtained using the zeroth-order expression for line-of-sight velocity given by Eq. (43). It is readily apparent that there will be some problems, because Ω' does not appear anywhere in this equation and because a and i appear only in the product form asi. Without knowledge of μ and subsequent use of Eq. (23), the best we can hope to do is to determine e, ω, t_p, n, and p_1. Note that E is related to e, t_p, n, and t by means of Kepler's equation

$$E - esE = n(t - t_p) \tag{50}$$

The partial derivatives of $v_{\rm los}^0$ with respect to ω and p_1 are readily verified to be

$$\frac{\partial v_{\rm los}^0}{\partial\omega} = -np_1 f_2(e,\omega,E) \tag{51}$$

$$f_2(e,\omega,E) = \frac{-c\omega sE - (1-e^2)^{1/2} s\omega cE}{1-ecE} \tag{52}$$

$$\frac{\partial v_{\rm los}^0}{\partial p_1} = -nf_1(e,\omega,E) \tag{53}$$

To evaluate the partial derivatives of $v_{\rm los}^0$ with respect to t_p and n, note that $v_{\rm los}^0$ depends on these quantities implicitly through E and that

$$\frac{\partial E}{\partial t_p} = -\frac{n}{1-ecE} \tag{54}$$

$$\frac{\partial E}{\partial n} = \frac{t-t_p}{1-ecE} \tag{55}$$

Next, note that

$$f_4(e,\omega,E) = \frac{\partial f_1/\partial E}{1-ecE} = \frac{-s\omega(cE-e) - (1-e^2)^{1/2} c\omega sE}{(1-ecE)^3} \tag{56}$$

We may now obtain the partial derivatives of $v_{\rm los}^0$ with respect to t_p and n as

$$\frac{\partial v_{\rm los}^0}{\partial t_p} = n^2 p_1 f_4 \tag{57}$$

$$\frac{\partial v_{\rm los}^0}{\partial n} = -p_1 f_1 - p_1 f_6 \tag{58}$$

$$f_6 = (E - esE) f_4 \tag{59}$$

Noting that $v_{\rm los}^0$ depends on e explicitly, as well as implicitly through E, and noting that

$$\frac{\partial E}{\partial e} = \frac{sE}{1-ecE} \tag{60}$$

we obtain

$$\begin{aligned} \frac{\partial v_{\rm los}^0}{\partial e} &= -np_1\left(f_8 + f_9 - \frac{ec\omega cE}{(1-e^2)^{1/2}(1-ecE)}\right) \\ &= -np_1\left(f_8 + f_9 - \frac{ec\omega(c\omega f_1 - s\omega f_2)}{1-e^2}\right) \end{aligned} \tag{61}$$

where

$$f_8(e,\omega,E) = sEf_4(e,\omega,E) \tag{62}$$

$$f_9(e,\omega,E) = cEf_1(e,\omega,E)/(1-ecE) \tag{63}$$

We are now faced with a problem expressible in general terms as that of estimating some constant vector $\boldsymbol{p}$ by suitable processing of noisy scalar measurements of the form

$$z(t) = h(\boldsymbol{p},t) + w(t) \tag{64}$$

where $w(t)$ is a white-noise process. If $\boldsymbol{p}^*$ is a nominal value for $\boldsymbol{p}$, and

$$z^*(t) = h(\boldsymbol{p}^*,t) \tag{65}$$

then to first order in $\Delta\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p}^*$,

$$\Delta z(t) = z(t) - z^*(t) = H(t)\Delta\boldsymbol{p} + w(t) \tag{66}$$

where

$$H(t) = \left[\frac{\partial h(\boldsymbol{p},t)}{\partial \boldsymbol{p}}\right]_{\boldsymbol{p}=\boldsymbol{p}^*} \tag{67}$$

If $P(t)$ denotes the error covariance matrix associated with the parameter vector p, based upon measurements through time t, then

$$P^{-1}(t)=P^{-1}(t_0)+\int_{t_0}^{t} H^T(\tau)H(\tau)q^{-1}(\tau)\mathrm{d}\tau \tag{68}$$

where $q(t)$ is the power spectral density associated with $w(t)$.

In our particular problem, H is more readily expressed in terms of E than t; the data may be thought of as being taken over one spacecraft orbit, with no a priori knowledge of the parameters to be estimated; and q may be regarded as constant in time. Thus, the inverse of the error covariance matrix after the orbit of data is given by

$$I=(1/nq)\int_{-\pi}^{\pi} H^T(E)H(E)(1-ecE)\mathrm{d}E \tag{69}$$

where use has been made of the relationship

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\frac{n}{1-ecE} \tag{70}$$

which follows from Eq. (50).

For the zeroth-order model of the Doppler data given by Eq. (43), the parameter vector p consists of e, ω, t_p, n, and p_1. The 5×5 information matrix given by Eq. (69) will be nonsingular if and only if the partial derivatives of $v^0_{\rm los}$ with respect to e, ω, t_p, n, and p_1, which comprise the elements of the row vector $H(E)$, are linearly independent functions of E over the interval $[-\pi,\pi]$.[20] By Eqs. (51), (53), (57), (58), and (61), this reduces to checking whether the functions f_1, f_2, f_4, f_6, and f_8+f_9 are linearly independent, assuming that $p_1\neq0$ From Eqs. (45), (52), (56), (59), (62), and (63), we can see that all five of these functions are linearly independent over the interval $[-\pi,\pi]$, at least for $e\neq0$. Thus, we conclude that all five of the parameters e, ω, t_p, n, and p_1 can be estimated from a single orbit of data, with no a priori information about these quantities, using the zeroth-order Doppler data model, as long as $p_1\neq0$ and $e\neq0$. It is stated in Ref. 6 that all orbital elements except Ω' are determinate, even when μ is unknown. However, we have argued here that a and i cannot be determined separately, using the zeroth-order model, if μ is unknown.

For simplicity, we have assumed that Doppler data are available over an entire spacecraft orbit. This is often not the case, due to passage of the spacecraft behind the body being orbited during part of the orbit. Such an outage of data complicates the mathematics but does not change the conclusions about linear independence of the partial derivatives. In addition, Doppler data are not really gathered continuously but are sampled in integrated form at discrete instants in time. For sampling intervals that are small compared to the orbital period, the continuous approximation is quite accurate.

Circular Orbits

When $e=0$, the functions f_2 and f_4 become identical, so that the partial derivatives of $v^0_{\rm los}$ with respect to ω and t_p become proportional. It is thus impossible to separate ω from t_p when $e=0$. In particular, we find that

$$\frac{\partial v^0_{\rm los}}{\partial\omega}+\frac{1}{n^*}\frac{\partial v^0_{\rm los}}{\partial t_p}=0 \tag{71}$$

where n^* is the nominal value of n. This means that the sum of quantities $\omega-n^*t_p$ can be determined, while the sum $\omega+n^*t_p$ is indeterminate. This is, of course, a fundamental property of a circular orbit, not a problem with the Doppler data.

Face-On Orbits

When $p_1=0$, corresponding to $i=0$ or 180 deg, only p_1 is well determined. The partial derivatives of $v^0_{\rm los}$ with respect to e, ω, t_p, and n all vanish, so that these quantities are indeterminate.

Central-Body Mass Known A Priori

It has been assumed to this point that μ, the gravitational parameter of the attracting planet, is completely unknown. If, on the other hand, this quantity is precisely known, from independent data, then a and n are precisely related according to Eq. (23). We may determine a and i from estimates of p_1 and n, provided that the Jacobian determinant

$$\frac{\partial(n,p_1)}{\partial(a,i)}=\begin{vmatrix}\dfrac{\partial n}{\partial a} & \dfrac{\partial n}{\partial i}\\[2ex] \dfrac{\partial p_1}{\partial a} & \dfrac{\partial p_1}{\partial i}\end{vmatrix}=\begin{vmatrix}-3n/2a & 0\\[2ex] si & aci\end{vmatrix}=-3nci/2 \tag{72}$$

is nonzero. Thus, when μ is precisely known, a and i can be estimated separately, except when $i=90$ deg, in which case a is well determined but i is indeterminate. In addition, a is indeterminate when $i=0$ or 180 deg, even though the Jacobian determinant is well behaved there, because n, as noted above, is indeterminate.

The conclusions obtained so far about indeterminate orbital elements are summarized in the upper halves of Tables 1 and 2.

Table 1 Indeterminate orbital elements with central-body mass unknown a priori[a]

Doppler data model/ Eccentricity	Inclination: $0<i<180$ deg	Inclination: $i=0$ or 180 deg
Zeroth-order model; $0<e<1$	Ω' indeterminate; a, i, μ known only through asi and μ/a^3	a, e, ω, Ω', t_p, and μ indeterminate
Zeroth-order model; $e=0$	Same as above, except ω and t_p known only through $\omega-n^*t_p$	Same as above
First-order model ($v_t\neq0$); $0<e<1$	Ω' and i indeterminate when $\Omega'=0$ or 180 deg and $i=90$ deg simultaneously	ω and Ω' known only through $\omega+\Omega'$ for $i=0$ deg, $\omega-\Omega'$ for $i=180$ deg
First-order model ($v_t\neq0$); $e=0$	a, i, Ω', μ known only through asi, $aci\Omega'$, and μ/a^3; ω and t_p known only through $\omega-n^*t_p$	ω, Ω', t_p known only through $\omega+\Omega'-n^*t_p$ for $i=0$ deg, $\omega-\Omega'-n^*t_p$ for $i=180$ deg

[a] Quantities of interest are a, e, i, ω, Ω', t_p, and μ. All are well determined, except as noted.

Table 2 **Indeterminate orbital elements with central-body mass known a priori**[a]

Doppler data model/ Eccentricity	Inclination	
	$0<i<180$ deg	$i=0$ or 180 deg
Zeroth-order model; $0<e<1$	Ω' indeterminate; i indeterminate when $i=90$ deg	a, e, ω, Ω', and t_p indeterminate
Zeroth-order model; $e=0$	Same as above, except ω and t_p known only through $\omega-n^*t_p$	Same as above
First-order model ($v_t\neq 0$); $0<e<1$	Ω' and i indeterminate when $\Omega'=0$ or 180 deg and $i=90$ deg simultaneously	ω and Ω' known only through $\omega+\Omega'$ for $i=0$ deg, $\omega-\Omega'$ for $i=180$ deg
First-order model ($v_t\neq 0$); $e=0$	Ω' indeterminate when $i=90$ deg or $\Omega'=90$ or 270 deg; i indeterminate when $\Omega'=0$ or 180 deg and $i=90$ deg simultaneously; ω and t_p known only through $\omega-n^*t_p$	ω, Ω', t_p known only through $\omega+\Omega'-n^*t_p$ for $i=0$ deg, $\omega-\Omega'-n^*t_p$ for $i=180$ deg

[a]Quantities of interest are a, e, i, ω, Ω', and t_p. All are well determined, except as noted.

Orbit Determination Using a First-Order Model for the Doppler Shift

Elliptical Orbits with Unknown Central-Body Mass

We shall now try to resolve some of the indeterminacies associated with the zeroth-order model of the line-of-sight velocity by examining the first-order terms in Eq. (46). We may rewrite that equation as

$$\begin{aligned} v^1_{\rm los} &= (np_1f_1+v_z)(p_1/R)f_4(1-ecE)^3+(na^2e/R)sE \\ &+\{[v_t(p_3c\omega-p_2s\omega)+v_zp_1s\omega](cE-e) \\ &+(1-e^2)^{1/2}[v_t(-p_3s\omega-p_2c\omega)+v_zp_1c\omega]sE\}/R \\ &+[(v_t/R)(-p_2f_1+p_3f_2) \\ &-(dv_z/dt)/n][(E-esE)+n(t_p-t_p^*)] \end{aligned} \tag{73}$$

where

$$p_2=acis\Omega' \tag{74}$$

$$p_3=ac\Omega' \tag{75}$$

Now let us assume that

$$nae\ll v_t \tag{76}$$

Normally, na (roughly the orbital speed of the spacecraft relative to the planet) is less than v_t but not enormously so. However, this assumption will simplify the mathematics somewhat, since the term in Eq. (73) involving a by itself (rather than embedded in p_1, p_2, or p_3) disappears.

We then obtain

$$\begin{aligned} v^1_{\rm los} &= (p_1/R)(np_1f_1+v_z)f_{12} \\ &+\{v_t[p_2(f_{12}-f_{13})+p_3(f_{11}+f_{14}) \\ &+(-p_2f_1+p_3f_2)n(t_p-t_p^*)]-v_zp_1f_{12}\}/R \\ &-(dv_z/dt)[(E-esE)/n+t_p-t_p^*] \end{aligned} \tag{77}$$

where

$$f_{11}(e,\omega,E)=c\omega(cE-e)-(1-e^2)^{1/2}s\omega sE \tag{78}$$

$$f_{12}(e,\omega,E)=f_4(1-ecE)^3 \tag{79}$$

$$f_{13}(e,\omega,E)=f_1(E-esE) \tag{80}$$

$$f_{14}(e,\omega,E)=f_2(E-esE) \tag{81}$$

Since p_2 and p_3 are the two parameters that are indeterminate in the zeroth-order model, they are of particular interest here. Noting that t_p is nominally equal to t_p^*, we find that

$$\frac{\partial v^1_{\rm los}}{\partial p_2}=\frac{v_t(f_{12}-f_{13})}{R} \tag{82}$$

$$\frac{\partial v^1_{\rm los}}{\partial p_3}=\frac{v_t(f_{11}+f_{14})}{R} \tag{83}$$

For $e\neq 0$, the functions f_{11}, f_{12}, f_{13}, and f_{14} are linearly independent of one another, and also of f_1, f_2, f_4, f_6, f_8, and f_9. Thus p_2 and p_3 can be determined using the first-order model, provided that $e\neq 0$ and $v_t\neq 0$. Determination of p_1, p_2, and p_3, however, does not always guarantee that a, i, and Ω' can be determined. We need to look at the Jacobian determinant

$$\frac{\partial(p_1,p_2,p_3)}{\partial(a,i,\Omega')}=\begin{vmatrix} \frac{\partial p_1}{\partial a} & \frac{\partial p_1}{\partial i} & \frac{\partial p_1}{\partial \Omega'} \\ \frac{\partial p_2}{\partial a} & \frac{\partial p_2}{\partial i} & \frac{\partial p_2}{\partial \Omega'} \\ \frac{\partial p_3}{\partial a} & \frac{\partial p_3}{\partial i} & \frac{\partial p_3}{\partial \Omega'} \end{vmatrix} = \begin{vmatrix} si & aci & 0 \\ cis\Omega' & -asis\Omega' & acic\Omega' \\ c\Omega' & 0 & -as\Omega' \end{vmatrix} = a^2(s^2is^2\Omega'+c^2i) \tag{84}$$

This determinant is nonzero except when $i=90$ deg and $\Omega'=0$ or 180 deg. When both i and Ω' take on these values, a is well determined but i and Ω' are indeterminate. Note that singular behavior occurs when $i=90$ and $\Omega'=0$ deg (or 180 deg) simultaneously, not simply when either of these conditions occurs alone. Thus, i can be determined when nominally 90 deg, from knowledge of p_2, as long as $s\Omega' \neq 0$. However, i can be determined better away from 90 deg, from knowledge of both p_1 and p_2, because p_1 can be determined using the zeroth-order Doppler data model, whereas determination of p_2 requires the first-order model and is therefore much less accurate. It is also clear that for i anywhere near 90 deg, i can be determined more accurately when Ω' is near 90 or 270 deg, than when Ω' is near 0 or 180 deg. It follows similarly that Ω' can be determined when nominally 0 or 180 deg from knowledge of p_2, as long as $ci \neq 0$. However, p_3 offers more information about Ω' than p_2 on the average, due to the ci term in the latter quantity. Thus, we would expect to be able to determine Ω' better when near 90 or 270 deg (from p_3) than when near 0 or 180 deg (from p_2). The closer i is to 90 deg, the more substantial the dependence of the accuracy of determining Ω' on the nominal value of that quantity. It would be wrong, however, to say that the first-order model has a true singularity at $\Omega' = 0$ or 180 deg. It would also be wrong to say that this model has a true singularity at $i = 90$ deg. Although both of these situations have sometimes been regarded as singular in the past,[21] they merely produce somewhat degraded accuracies. The true singularity of this model occurs when both conditions are satisfied simultaneously. The conclusions obtained here are fully consistent with the numerical results presented in Ref. 8, though not with the way those results have sometimes been interpreted.

These conclusions are also borne out by flight data in the case of the Pioneer Venus Orbiter mission. The condition $\Omega' = 0$ deg was approximately fulfilled during orbit 84, and the condition $i = 90$ deg was approximately fulfilled during orbit 159. In neither case did the expected serious degradation in orbit determination accuracy occur.[4] In the former case, i was not near 90 deg, and essentially no growth in orbit determination errors was observed. In the latter case, Ω' was not near 0 or 180 deg, and even though the uncertainty in i grew substantially, it was still only .0005 deg.[4] The latter growth can be attributed to the fact that the first-order model was needed to determine i near $i = 90$ deg, while the zeroth-order model was adequate (with a priori knowledge of the mass of Venus) for $i \neq 90$ deg.

In summary, some rotation of the plane-of-sky coordinate system (i.e., $v_t \neq 0$) is needed in order to be able to determine p_2 and p_3. Knowledge of p_2 and p_3 allows i and Ω' to be inferred, except when the orbit is viewed edge on ($i = 90$ deg) and that edge-on geometry is not changing ($di/dt = 0$ or, equivalently, $s\Omega' = 0$).

Circular Orbits with Unknown Central-Body Mass

Now, let us investigate what happens when $e = 0$. Let us denote by f_i^e the function f_i when $e = 0$. We find that

$$f_1^e(\omega,E) = -s\omega sE + c\omega cE = f_{11}^e(\omega,E) \tag{85}$$

$$f_2^e(\omega,E) = -c\omega sE - s\omega cE = f_4^e(\omega,E) = f_{12}^e(\omega,E) \tag{86}$$

$$f_6^e(\omega,E) = E(-c\omega sE - s\omega cE) = f_{14}^e(\omega,E) \tag{87}$$

$$f_8^e(\omega,E) = sE(-c\omega sE - s\omega cE) \tag{88}$$

$$f_9^e(\omega,E) = cE(-s\omega sE + c\omega cE) \tag{89}$$

$$f_{13}^e(\omega,E) = E(-s\omega sE + c\omega cE) \tag{90}$$

We note that f_{11}, f_{12}, and f_{14} degenerate into f_1, f_2, and f_6, respectively, for $e = 0$. From Eqs. (58) and (83) we see that

$$\frac{\partial v_{\rm los}^0/\partial n}{p_1} + \frac{R(\partial v_{\rm los}^1/\partial p_3)}{v_t} = 0 \tag{91}$$

for $e = 0$, so that p_3 affects the first-order model in the same way that n affects the zeroth-order model. Thus, we will not be able to determine p_3 independently of n with the first-order model for $e = 0$. If one differentiates $v_{\rm los}^1$ with respect to n, one finds that the resulting partial derivative contains some functional dependencies on E that differ from those in $f_1, f_2, f_4, f_6, f_8, f_9, f_{11}, \ldots, f_{14}$. However, this is only as helpful in determining p_3 as the use of terms involving p_3 in a second-order Doppler data model. Such a model will not be explored here.

Thus, we find that for $e = 0$, p_1 and p_2 can be determined with a first-order data model (in fact, p_1 can be determined from a zeroth-order model), but p_3 cannot be so determined. Thus we can determine three nonlinear combinations (n, p_1, and p_2) of the four parameters, μ, a, i, and Ω' but cannot, in general, determine any of these parameters individually.

Central-Body Mass Known A Priori

So far in this section, the gravitational parameter μ has been assumed to be completely unknown prior to the start of the data arc. If, on the other hand, μ is known precisely in advance, then a and i can be determined from zeroth-order terms, provided that $i \neq 90$ deg, as has been noted previously. Determination of p_2 from first-order terms then allows determination of Ω', provided again that $i \neq 90$ deg and that Ω' is not 90 or 270 deg. We may confirm this by evaluating one more Jacobian determinant:

$$\frac{\partial(n,p_1,p_2)}{\partial(a,i,\Omega')} = \begin{vmatrix} \frac{\partial n}{\partial a} & \frac{\partial n}{\partial i} & \frac{\partial n}{\partial \Omega'} \\ \frac{\partial p_1}{\partial a} & \frac{\partial p_1}{\partial i} & \frac{\partial p_1}{\partial \Omega'} \\ \frac{\partial p_2}{\partial a} & \frac{\partial p_2}{\partial i} & \frac{\partial p_2}{\partial \Omega'} \end{vmatrix} = \begin{vmatrix} -3n/2a & 0 & 0 \\ si & aci & 0 \\ cis\Omega' & -asis\Omega' & acic\Omega' \end{vmatrix} = -3nac^2ic\Omega'/2 \tag{92}$$

which is nonzero except for $i = 90$ deg or $\Omega' = 90$ or 270 deg.

Thus, with the first-order Doppler data model, assuming that $i \neq 0$ or 180 deg (situations we have not yet investigated), all orbit elements, as well as μ, can be determined, as long as Ω' is not 0 or 180 deg, with $i = 90$ deg simultaneously, and as long as $e \neq 0$ and $v_t \neq 0$. If μ is known a priori, this same Doppler data model can be used to determine all meaningful elements of a circular orbit, as long as Ω' is not 90 or 270 deg, $i \neq 90$ deg, and $v_t \neq 0$, it again being assumed that $i \neq 0$ or 180 deg, since we have not yet investigated these situations. We see that the meaningful elements of a circular orbit are more difficult to determine than those of a more general elliptical orbit, a priori knowledge of μ being needed in the former case, at least as far as the first-order model is concerned. In the case of a circular orbiter, some information is available about $s\Omega'$ (through p_2), but no information is available about $c\Omega'$ (through p_3). Consequently, Ω' can be determined more accurately near 0 or 180 deg than near 90 or 270 deg. For i near 90 deg, the determination of $s\Omega'$ from p_2 degrades. Thus, we have great difficulty in determining Ω' when *either* Ω' is near 90 (or 270 deg) *or* i is near 90 deg. By contrast, in the case of $e \neq 0$, we have great difficulty in determining Ω' when *both* Ω' is near 0 deg (or 180 deg) *and* i is near 90 deg. Note that the problematical situations for $e \neq 0$ do not disappear when $e = 0$. What happens is that new problems appear that are even worse.

The conclusions reached here for the case of circular orbits are consistent with the analytical expression for the uncertainty in Ω' derived in Ref. 11. (See comments below on the validity of this expression when $i=0$ or 180 deg.) These conclusions are also consistent with the observation in Ref. 8 that the uncertainties in all orbital elements increase rapidly as e is reduced toward zero. The conclusions reached so far in this section are summarized in the lower halves of Tables 1 and 2.

The Mariner 9, Viking Orbiter, and Pioneer Venus Orbiter missions have all involved orbital eccentricities of at least 0.6. Thus, the low-eccentricity problems discussed here have not yet been encountered in a planetary flight project. A number of proposed planetary missions, however, beginning with the Mars Observer mission to be launched in 1990, involve near-circular orbits.[15,16] It should be noted that while the subject of this paper is two-way coherent Doppler data, this is not the only data type that can be used for orbit-determination purposes in a planetary orbiter mission. Differenced Doppler data (involving two receiving stations on the Earth) and narrow-band differential very-long-baseline interferometry (ΔVLBI) can be used to determine those orbital elements that are determined poorly using conventional Doppler data alone.[12-14,16]

It should also be noted at this point that the information matrix in Eq. (69) has been constructed based on a linearized version, Eq. (66), of the original measurement relationship given by Eq. (64). Thus our tests for determinancy of orbital elements are only local, rather than global, tests. It is noted in Ref. 6 that ambiguities will exist in determining the quadrants in which i and Ω' lie. This may be seen by inspection of Eqs. (44), (74), and (75). Although the Jacobian determinant relating p_1, p_2, and p_3 to a, i, and Ω' is generally nonzero [see Eq. (84)], knowledge of p_1, p_2, and p_3 leaves a correlated sign ambiguity in ci and $s\Omega'$.

Face-On Elliptical Orbits

Next, let us recall that only p_1 can be determined when $i=0$ or 180 deg, using the zeroth-order Doppler data model, and let us see if the first-order model can be used to determine the remaining quantities. Let us denote by $v_{\rm los}^{1i}$ the expression for $v_{\rm los}^{1}$ when $i=0$ or 180 deg. We obtain from Eq. (77)

$$v_{\rm los}^{1i}=\frac{v_t[p_2(f_{12}-f_{13})+p_3(f_{11}+f_{14})+(-p_2f_1+p_3f_2)n(t_p-t_p^*)]}{R}$$

$$-\frac{{\rm d}v_z}{{\rm d}t}(1-ecE)/n \quad +t_p-t_p^* \tag{93}$$

Let us evaluate first the partial derivative of $v_{\rm los}^{1i}$ with respect to ω. We obtain

$$\frac{\partial v_{\rm los}^{1i}}{\partial\omega}=v_t\frac{[(p_2(-f_{11}-f_{14})+p_3(f_{12}-f_{13})]}{R} \tag{94}$$

From Eqs. (82), (83), and (94), we have

$$\frac{\partial v_{\rm los}^{1i}}{\partial\omega}=p_3\frac{\partial v_{\rm los}^{1i}}{\partial p_2}-p_2\frac{\partial v_{\rm los}^{1i}}{\partial p_3} \tag{95}$$

so that ω cannot be determined independently of p_2 and p_3. To interpret this result more readily, note that for $i=0$ deg

$$\frac{\partial v_{\rm los}^{1i}}{\partial\Omega'}=\frac{\partial v_{\rm los}^{1i}}{\partial p_2}\frac{\partial p_2}{\partial\Omega'}+\frac{\partial v_{\rm los}^{1i}}{\partial p_3}\frac{\partial p_3}{\partial\Omega'}$$

$$=p_3\frac{\partial v_{\rm los}^{1i}}{\partial p_2}-p_2\frac{\partial v_{\rm los}^{1i}}{\partial p_3}=\frac{\partial v_{\rm los}^{1i}}{\partial\omega} \tag{96}$$

If we define

$$\omega_+=\omega+\Omega' \tag{97}$$

$$\omega_-=\omega-\Omega' \tag{98}$$

we find that

$$\frac{\partial v_{\rm los}^{1i}}{\partial\omega_+}=\left(\frac{\partial v_{\rm los}^{1i}}{\partial\omega}+\frac{\partial v_{\rm los}^{1i}}{\partial\Omega'}\right)\Big/2 \tag{99}$$

$$\frac{\partial v_{\rm los}^{1i}}{\partial\omega_-}=\left(\frac{\partial v_{\rm los}^{1i}}{\partial\omega}-\frac{\partial v_{\rm los}^{1i}}{\partial\Omega'}\right)\Big/2=0 \tag{100}$$

Thus, the first-order model is able to determine $\omega+\Omega'$, but not $\omega-\Omega'$. Note that ω and Ω' are rotations about the same axis when $i=0$ deg. By similar arguments, the first-order model can be shown able to determine $\omega-\Omega'$, but not $\omega+\Omega'$, for $i=180$ deg. (In this case, ω and Ω' are rotations about antiparallel axes.)

Next, note that

$$\frac{\partial v_{\rm los}^{1i}}{\partial a}=\frac{\partial v_{\rm los}^{1i}}{\partial p_2}\frac{\partial p_2}{\partial a}+\frac{\partial v_{\rm los}^{1i}}{\partial p_3}\frac{\partial p_3}{\partial a}$$

$$=v_t\frac{[p_2(f_{12}-f_{13})+p_3(f_{11}+f_{14})]}{aR} \tag{101}$$

Differentiation of $v_{\rm los}^{1i}$ with respect to E yields

$$\frac{\partial v_{\rm los}^{1i}}{\partial E}=v_t(1-ecE)\cdot$$

$$\frac{p_2[-2f_1-f_4(E-esE)]+p_3[2f_2-f_{11}(E-esE)/(1-ecE)^3]}{R} \tag{102}$$

With the use of Eqs. (54) and (55), we now obtain

$$\frac{\partial v_{\rm los}^{1i}}{\partial t_p}=-nv_t\frac{p_2(-f_1-f_6)+p_3(f_2-f_5)}{R} \tag{103}$$

$$\frac{\partial v_{\rm los}^{1i}}{\partial n}=v_t\frac{p_2(-2f_{13}-f_{16})+p_3(2f_{14}-f_{15})}{nR} \tag{104}$$

where

$$f_5(e,\omega,E)=f_{11}(E-esE)/(1-ecE)^3 \tag{105}$$

$$f_{15}(e,\omega,E)=(E-esE)f_5 \tag{106}$$

$$f_{16}(e,\omega,E)=(E-esE)^2f_4 \tag{107}$$

If we evaluate the effects of e on $v_{\rm los}^{1i}$, we obtain

$$\frac{\partial v_{\rm los}^{1i}}{\partial e}=v_t\frac{p_2(f_{17}-f_{21}+f_{19})+p_3(-f_{18}+f_{22}-f_{20})}{R} \tag{108}$$

$$+\text{other terms}$$

where

$$f_{17}=s\omega+ec\omega sE/(1-e^2)^{1/2} \tag{109}$$

$$f_{18}=c\omega-es\omega sE/(1-e^2)^{1/2} \tag{110}$$

$$f_{19}=sEf_1 \tag{111}$$

$$f_{20}=sEf_2 \tag{112}$$

$$f_{21}=(E-esE)(f_8+f_9)-ec\omega(c\omega f_{13}-s\omega f_{14})/(1-e^2) \tag{113}$$

$$f_{22}=(E-esE)(-f_7+f_{10})+es\omega(-s\omega f_{14}+c\omega f_{13})/(1-e^2) \tag{114}$$

$$f_7=sEf_3 \tag{115}$$

$$f_{10}=cEf_2/(1-ecE) \tag{116}$$

(The functions $f_1,\ldots, f_{22}$ have been numbered such that each even-numbered function is the partial derivative with respect to ω of the preceding odd-numbered function.)

To be able to determine a, e, t_p, n, and either $\omega+\Omega'$ or $\omega-\Omega'$, according to whether $i=0$ or 180 deg, in addition to the well-determined quantity i, it is necessary that the partial derivatives of v_{los}^{1i} with respect to these quantities contain terms independent of f_1 (involved in $\partial v_{\rm los}^0/\partial i$) and that these partial derivatives be independent of each other. This can be seen to be the case, at least for $e\neq 0$. Thus, the first-order model allows the determination of all meaningful elements of an elliptical orbit even when $i=0$ or 180 deg, assuming that $v_t\neq 0$.

Face-On Circular Orbits

The final situation to be investigated is that of a circular orbit when $i=0$ or 180 deg. Using the same notation as in Eqs. (85-90), we find that

$$f_5^e(\omega,E)=E(c\omega cE-s\omega sE)=f_{13}^e(\omega,E) \tag{117}$$

$$f_7^e(\omega,E)=sE(c\omega cE-s\omega sE)=f_{19}^e(\omega,E) \tag{118}$$

$$f_{10}^e(\omega,E)=cE(-c\omega sE-s\omega cE) \tag{119}$$

$$f_{15}^e(\omega,E)=E^2(c\omega cE-s\omega sE) \tag{120}$$

$$f_{16}^e(\omega,E)=E^2(-s\omega cE-c\omega sE) \tag{121}$$

$$f_{17}^e(\omega,E)=s\omega \tag{122}$$

$$f_{18}^e(\omega,E)=c\omega \tag{123}$$

$$f_{20}^e(\omega,E)=sE(-c\omega sE-s\omega cE)=f_8^e(\omega,E) \tag{124}$$

$$f_{21}^e(\omega,E)=E(f_8^e+f_9^e) \tag{125}$$

$$f_{22}^e(\omega,E)=E(-f_7^e+f_{10}^e) \tag{126}$$

We observe that for $e=0$ and $i=0$ deg,

$$\frac{\partial v_{\rm los}^{1i}}{\partial\omega_+}+\frac{1}{n^*}\frac{\partial v_{\rm los}^{1i}}{\partial t_p}=0 \tag{127}$$

by virtue of the fact that the functions f_{11}, f_{12}, f_{13}, and f_{14} reduce to f_1, f_2, f_5, and f_6. Thus, the quantity $\omega+\Omega'-n^*t_p$ can be determined, but $\omega+\Omega'+n^*t_p$ is indeterminate. By similar arguments, for $e=0$ and $i=180$ deg, $\omega-\Omega'-n^*t_p$ can be determined but $\omega-\Omega'+n^*t_p$ is indeterminate. These results are, of course, inevitable properties of a circular orbit viewed face on, not problems associated with the Doppler data. The analytical expression in Ref. 11 for the uncertainty in Ω' appears to be invalid near $i=0$ or 180 deg, since a finite uncertainty is predicted.

We find that the partial derivative of $v_{\rm los}^0$ with respect to i and the partial derivatives of $v_{\rm los}^{1i}$ with respect to a, n, e, and either $\omega+\Omega'-n^*t_p$ or $\omega-\Omega'-n^*t_p$, according to whether $i=0$ or 180 deg, are linearly independent, for $v_t\neq 0$. Thus, the parameters i, a, n, e and either $\omega+\Omega'-n^*t_p$ or $\omega-\Omega'-n^*t_p$ can all be determined.

Conclusions

The problem of determining the orbit of a spacecraft about a natural body other than the Earth or the sun by means of Doppler tracking data has been investigated analytically. Partial derivatives of a truncated power-series expansion of the spacecraft-tracking station line-of-sight velocity have been evaluated with respect to classical orbital elements and the central-body mass. Linear dependencies among these partial derivatives are used to indicate orbital elements (or combinations thereof) that are indeterminate. The indeterminate sets of elements are found to vary substantially according to whether the orbit is elliptical or circular, whether the orbit is viewed at an oblique angle or face on, and whether the central-body mass is known or unknown. The results obtained are considerably more general than, but for the most part consistent with, numerical and analytical results obtained in several previous studies. In general, with the use of two-way coherent Doppler data, it has been found that semimajor axis, eccentricity, time of periapsis passage, argument of periapsis (relative to the plane of the sky), and central-body mass (when unknown) are most readily determined at high inclination angles (relative to the plane of the sky). Inclination (relative to the plane of the sky) is most readily determined at low inclination angles. All quantities are more readily determined for high orbital eccentricities than for low eccentricities. Several geometries sometimes categorized as singular have been shown to be misclassified in that they produce only modest degradations in accuracy.

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